Journal for Geometry and Graphics
Volume 16 (2012), No. 1, 77–85.

Elastic Deformations in Polyhedral Rings Formed by Corpuscle Elements

Eva Wohlleben[1], **Wolfram Liebermeister**[2]

[1] *Muthesius-Kunsthochschule Kiel, Germany*
email: eva_wohlleben@yahoo.de; URL `www.korpuskel.de`

[2] *Charité – Universitätsmedizin Berlin, Germany*
email: wolfram.liebermeister@charite.de

Abstract. A variety of three-dimensional geometric structures can be formed by regular triangles that are flexibly connected along their edges. Polyhedra, chains, closed rings, and spatial networks can be built by interlinking building blocks called "corpuscles", each composed of a small number of triangles. Due to the flexible edges, some of the resulting structures may show collective movements. Open-ended corpuscle chains are fully flexible, while most closed structures can only move if the triangle edge lengths can be slightly deformed. Paper models indicate that closed, ring-like chains are most flexible if the number of elements in the ring is a multiple of three. To study this flexibility in detail, we simulate various corpuscle structures with edges modelled as elastic springs. First, the structures are relaxed towards conformations of minimal edge tension; then, elastic deformations are studied by inspecting the normal modes of the stress matrix. The simulations confirm that most of the corpuscle rings can only be closed if some tension is applied. When the tension is high, some corpuscle elements are spontaneously punched in and the structure loses its symmetry. In the two structures that show a strong symmetry breaking, the ring size is a multiple of three, as suggested by the previous studies of open-ended chains.

Key Words: Corpuscle, flexible polyhedron, symmetry breaking, elastic deformation.

MSC 2010: 51P05, 74B05

1. Introduction

Regular triangles, connected by common edges, can form a variety of three-dimensional geometric structures. In [2], we have shown how polyhedra, polyhedral rings and networks can be built from basic elements, called *"corpuscles"*, each consisting of several triangles in different arrangements. As their name says, these "little bodies" are building blocks, not autonomous structures. Figure 1 (a) shows a corpuscle element consisting of ten regular triangles in the

ISSN 1433-8157/$ 2.50

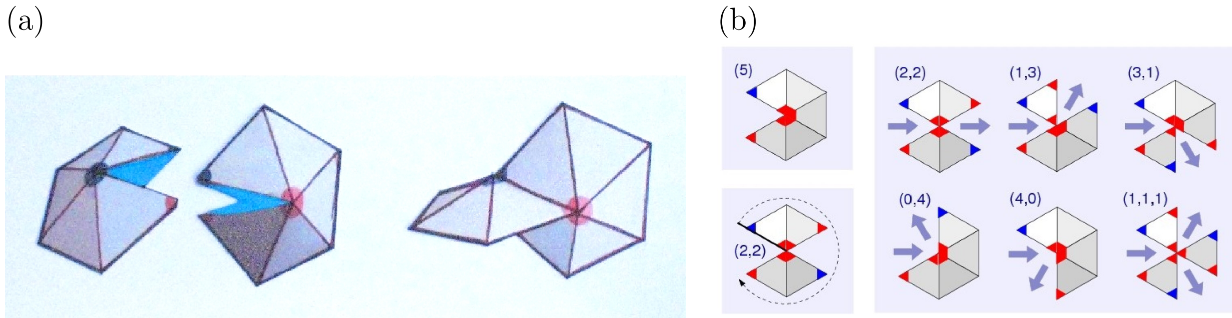

Figure 1: Corpuscles and the connection between them. (a) The Goldberg icosahedron [1] can be built by connecting two open corpuscles mouth to mouth. (b) Top left: corpuscle element in flat conformation, seen from top. Bottom left: to build chains, one of the existing segments is replaced by a second mouth. Our short notation for types of corpuscles works as follows: we look from the top, start at the thick line, and describe the groups of connected segments in clockwise direction order. In this case, we obtain two segments, a mouth, and two more segments, or briefly, (2,2). Right: all five corpuscle types with four segments and two mouths, as well as the branch point type (1,1,1) with three segments and three mouths. Arrows indicate how corpuscle chains can be extended: corpuscles are attached to an existing chain with their left mouth (incoming arrow).

following arrangement: five triangles form a pentagonal pyramid, and another such pyramid is joined to it bottom to bottom. A triangle on top side and its counterpart on the bottom are called a *"segment"*. As long as all ten triangles are connected, the overall structure is rigid. However, if we cut the corpuscle along the edges between two of the segments, the slit creates an open end *("mouth")*. Since the edges between triangles are flexible, the angles between adjacent triangles can change, and we can deform the entire structure from the original double pyramid to a double layered, flat hexagon in which one of the triangles is missing.

Single corpuscles can be voluminous or flat and show their characteristics only when they are part of larger compounds. Since the connecting edges act like hinges, some of the resulting structures are flexible, showing global conformation changes with little or no distortion of the triangle edge lengths. Two corpuscle elements, for instance, can be joined through their mouths. The result is GOLDBERG's *"Siamese dipyramid" icosahedron* [1], which can show collective movements: if one element has a bold shape (long central axis, narrow mouth), the other one is flat (short axis, wide mouth). The movement from one position to the other requires a deformation of the edge lengths, but this deformation is so small that paper models appear fully flexible.

More complex structures can be built from other types of corpuscle elements [2] as shown in Figure 1(b). Segments and mouths are always arranged around a central axis. The elements can be denoted by listing the numbers of neighbouring segments: for example, a (2,2) corpuscle element contains two pairs of segments, separated by two mouths; a (1,3) element contains a single segment, a mouth, three segments, and another mouth. Other types of elements are denoted accordingly. Corpuscles can be connected to build a variety of structures. Several corpuscles with two mouths can form chains or closed rings, while elements with three or more mouths can serve as branch points in extended networks [2].

2. Closed corpuscle structures and their possible deformations

Paper models indicate that open-ended corpuscle chains can be flexibly deformed and that their deformations are approximately periodic along the chain, with a period around three. This approximate periodicity has been confirmed by previous calculations [2]. Accordingly, corpuscle rings made from paper are deformable if the number of elements is a multiple of three, while other rings are rigid. The 8-ring, for instance, is rigid, while a 12-ring can be deformed easily, showing a repeated sequence of flat/bold/bold elements. Moreover, the paper model of a new 6-ring flips spontaneously into a non-symmetric conformation, which apparently reduces the tension in the material. However, paper models do not directly teach us which rings can be built from exact regular triangles, how strongly the edges need to be deformed to close the other rings, and which edges make the structures resist further deformations.

To study this, we analyse a number of corpuscle structures including rings consisting of 6, 8, 10, 12, and 60 segments and simulate their stable conformations and elastic deformations. In our numerical model, edges are represented by elastic springs following Hooke's law. This type of model is common in molecular dynamics and in the calculation of force-directed graph layouts [3]. We check how much the edges have to be deformed to obtain the closed structures and whether the stable conformations, in which the edges show a minimal tension, show a spontaneous symmetry breaking.

Having computed these stable conformations, we study how strongly they resist further deformations. Small elastic deformations can be characterised by stress energies, which depend quadratically on node displacements. The eigenvectors of the quadratic form, called deformation modes, represent natural deformations of the structure, similar to the natural harmonics of guitar strings. Six of these displacement modes correspond to simple translations and rotations in space and require no deformation energy. All other modes represent elastic deformations and are associated with positive energies. The corresponding forces tend to drive the structure back towards their stable conformation, and if the nodes carry identical masses, this dynamic will lead to global oscillations of the structure. Modes associated with large deformation energies *("hard modes")* will show fast oscillations; modes with small deformation energies *("soft modes")* will show a slower "breathing".

3. Elastic spring model: stable conformations and deformation modes

Aside from the structures described in [2], we consider here a 44-hedral cluster, another cluster of four elements, and new rings composed of 6, 12, or 60 elements. Together with the Goldberg icosahedron, our list thus comprises corpuscle clusters and rings of 2, 4, 6, 8, 12, 16, and 60 elements, as well as the cube-centric corpuscle ball (20 elements) and the 44-hedral cluster. All structures are shown in Fig. 2 as paper models and in Fig. 4 as computer models. Their topologies are listed in Table 1. Most structures can only be closed if the edges are elastic and can therefore change their length. We have studied this by a numerical model in which edges are elastic springs following Hooke's law. Conformations were scored by energies resulting from stretching and compression of the edges and for each structure, we computed a conformation of minimal total edge tension ("stable conformation") and determined its symmetry. Next, we determined possible low-energy deformations around this stable conformation from the eigenvectors of the energy function's Hessian matrix.

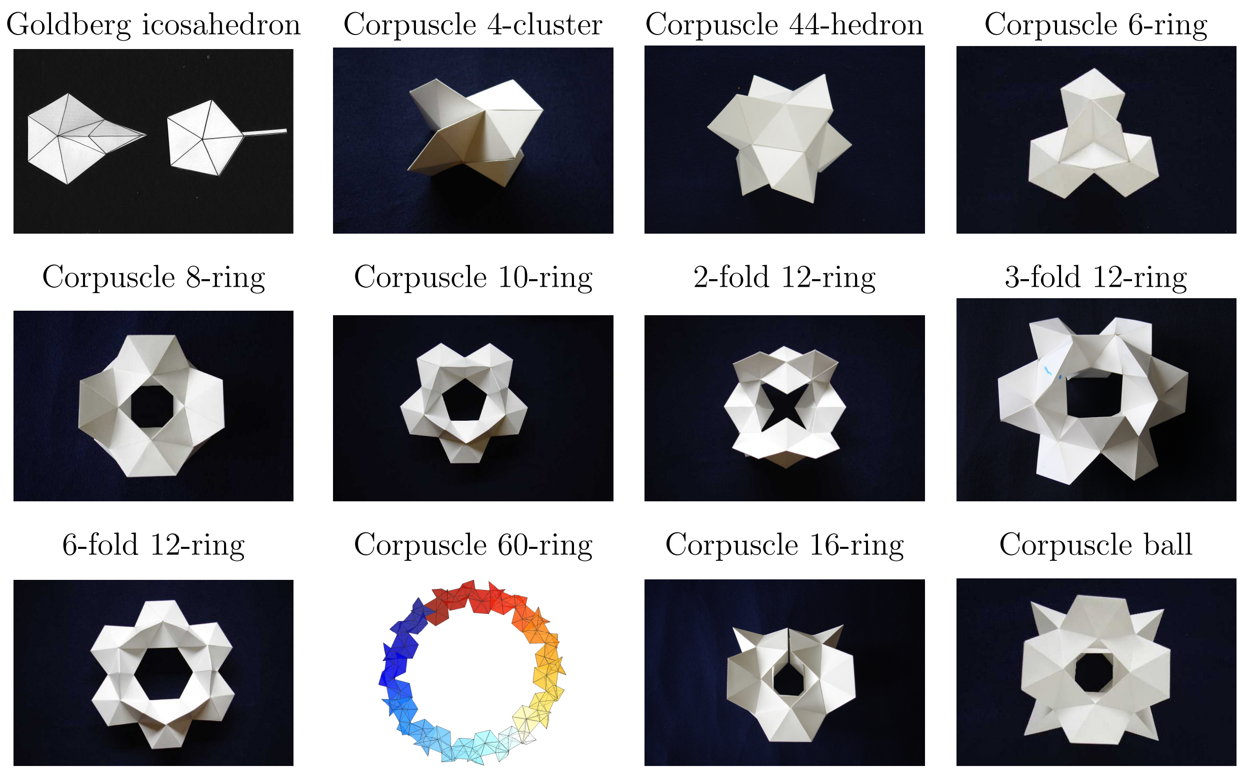

Figure 2: Corpuscle structures shown as paper models. For details, see Table 1.

In the calculations, a corpuscle structure was represented by a set of nodes — with indices α and coordinate vectors $(x_{1\alpha}, x_{2\alpha}, x_{3\alpha})^T$ — and by edges given as ordered pairs of nodes. The total energy reads

$$E = \tfrac{1}{2} \sum_{(\alpha,\beta)} (D_{\alpha\beta} - L_{\alpha\beta})^2,$$

where $D_{\alpha\beta}$ is the Euclidean distance between nodes with indices α and β, and $L_{\alpha\beta}$ is the nominal edge length between them. Usually, we consider a nominal edge length $L_{\alpha\beta} = 1$. The sum runs over all pairs (α, β) of nodes joined by an edge and satisfying $\alpha < \beta$. To obtain closed rings, we started from open chains, determined nodes at the open ends to be matched, and deformed the structures such as to bring these nodes in close vicinity. Then, we collapsed the matched nodes and further relaxed the structure numerically to obtain a stable conformation, that is, a local energy minimum. Whenever all edge lengths in this conformation were equal to 1 within numerical accuracy, we concluded that the structure can be built exactly without edge deformation.

Next, we studied the soft deformation modes. Based on the stable conformation obtained, we computed the Hessian matrix of the energy function and determined its eigenvalues and eigenvectors. For these calculations, we made two alternative assumptions:

(i) all edges have the same nominal length, i.e., they may already be stretched or compressed in the stable conformation;

(ii) each edge's nominal length is defined by the length in the stable conformation, i.e., different edges have different nominal lengths, and the stable conformation is free of tension by construction.

The results were similar, but the second alternative excludes negative eigenvalues that could

Table 1: Corpuscle structures studied (for a detailed description, see the Appendix and compare Figs. 3 and 4)

Name	*Elements*	*Nodes*	*Edges*	*Triangles*	*Rotation symmetry*	*Mean edge energy*	*Symmetry breaking*	*Exact*
Goldberg icosahedron	2	12	30	20	2	≈ 0		$\times$
Corpuscle 4-cluster	4	18	48	32	2	≈ 0		$\times$
44-hedral cluster	6	24	66	44	2, 3	≈ 0		$\times$
Corpuscle 6-ring	6	25	72	48	3	≈ 0	$\times$	
Corpuscle 8-ring	8	32	96	64	4	≈ 0		$\times$
Corpuscle 10-ring	10	40	120	80	5	$8.5 \cdot 10^{-5}$		$\times$
Corpuscle 12-ring (2×)	12	48	144	96	2	$6.4 \cdot 10^{-5}$		
Corpuscle 12-ring (3×)	12	48	144	96	3	$3.9 \cdot 10^{-5}$		
Corpuscle 12-ring (6×)	12	48	144	96	6	$7.6 \cdot 10^{-8}$	$\times$	
Corpuscle 60-ring	60	240	720	480	10	$3.1 \cdot 10^{-8}$		
Corpuscle 16-ring	16	56	188	128	2	$3.3 \cdot 10^{-5}$		
Corpuscle ball	20	64	216	144	4	$2.4 \cdot 10^{-5}$		

otherwise be caused by numerical inaccuracies. In the following, we describe the results obtained from the second alternative. In any case, translation and rotation modes of the entire structure were energy-neutral and their zero eigenvalues were recovered with good numerical accuracy. The next smallest eigenvalues are associated with soft elastic deformations.

4. Symmetry and elastic deformations in clusters and rings

Paper models and simulation results for the relaxed conformations are shown in Figs. 2 and 4; details are listed in Table 1. In the table, the first columns summarise the numbers of elements, nodes, edges, and faces. The first three structures ("clusters") satisfy Euler's formula for convex polyhedra (the numbers of nodes and faces together equals the number of edges plus 2), while all other structures, due to their differing topologies, have Euler's characteristics different from 2. "Rotation symmetry" refers to an idealised geometric shape of maximal symmetry, which may show unequal edge lengths and be unstable in the elastic spring model.

The last three columns summarise results about the relaxed state, obtained from the numerical optimisation. Average edge tensions (mean square deviation from the nominal edge length of 1) were computed after relaxing the structure to a stable conformation (compare Fig. 3). Values below 10^{-9} are labelled as "≈ 0"; larger values suggest that the structure cannot be formed with equal edge lengths. The column "symmetry breaking" indicates whether the stable conformation breaks the symmetry of the graph (as determined by visual inspection and by multiplicities of edge lengths). A structure is labelled "exact" if there exists a symmetric stable conformation with equal edge lengths (again, within numerical accuracy). Paper models and simulation results for elastic deformations are shown in Figs. 2 and 4. The structures' shapes and behaviours are described in more detail in Appendix A, and movies showing soft simulated deformation modes can be found at `www.korpuskel.de`.

To summarise, our computer simulations show that the three clusters (Goldberg icosahedron, 4-cluster, and 44-hedron) can be built exactly with regular edge lengths, while the rings

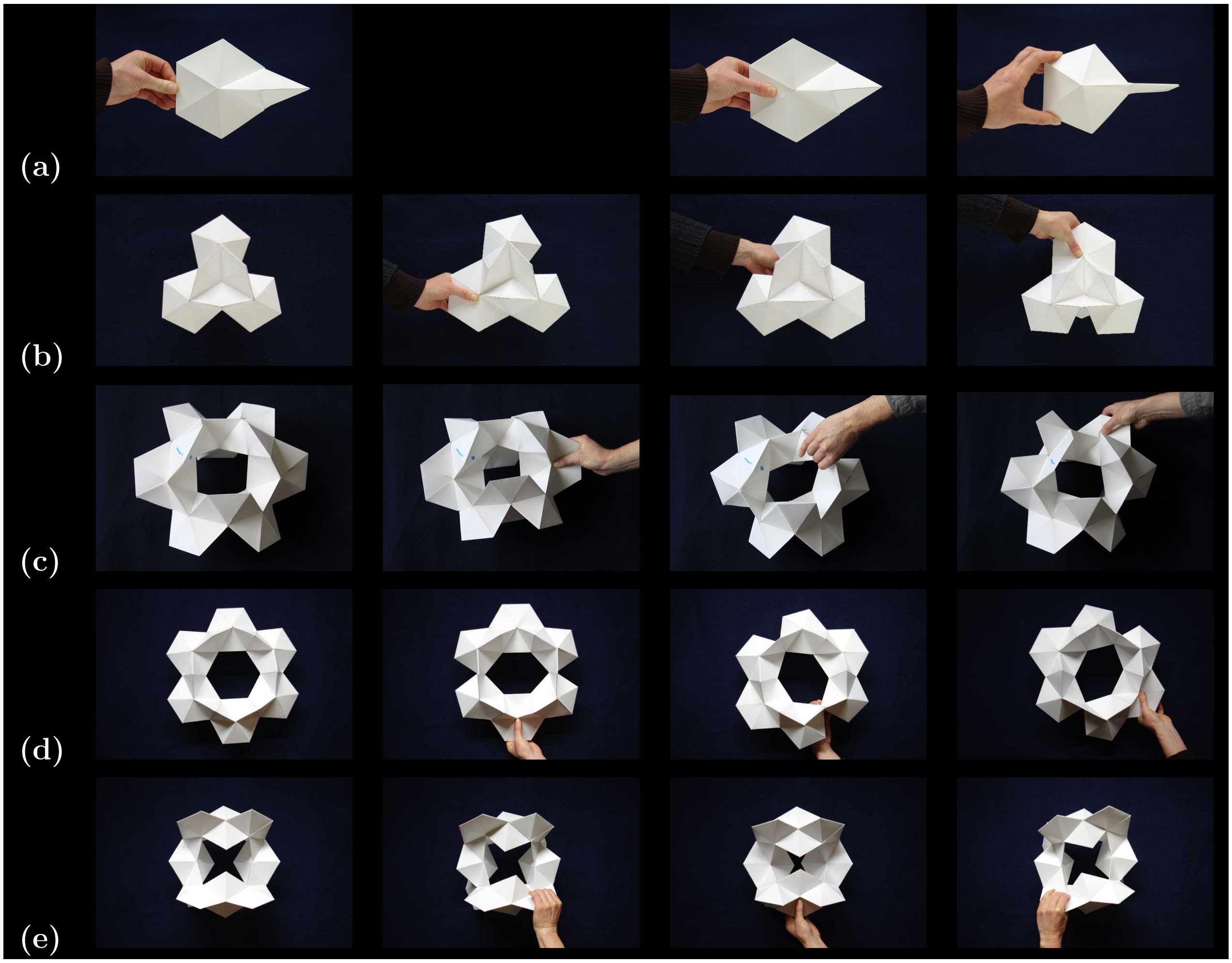

Figure 3: Deformation of corpuscle compounds. The symmetric conformations are shown on the left, the extreme deformations in the other columns. (a) Paper model of the Goldberg icosahedron with its two extreme conformations. (b) 6-ring. (c) 3-fold symmetric 12-ring. (d) 6-fold symmetric 12-ring. The rings shown in (b)–(d) have three extreme deformations. For symmetry reasons, these conformations are identical except for rotations of the structure. (e) Symmetric conformation and the three extreme deformations of the 2-fold symmetric 12-ring. In one extreme conformation (mid-right), all (2,2) type elements are flat. The other two extreme conformations (mid-left; right) are mirror images of each other.

can only be closed by applying some tension — the 8-ring being the only exception. If the overall tension is high, it will not be distributed evenly over the entire structure, but some of the corpuscle elements may become flattened or even punched in spontaneously, breaking the overall symmetry. This happens in the 6-ring, in the 12-ring with 6-fold symmetry, and in other rings with stronger tension (18-ring and 24-ring) which were not presented here. Although our selection of corpuscle structures is far from being comprehensive, the results so far agree with our expectation that flexible movements and deformations are facilitated if ring sizes are multiples of three.

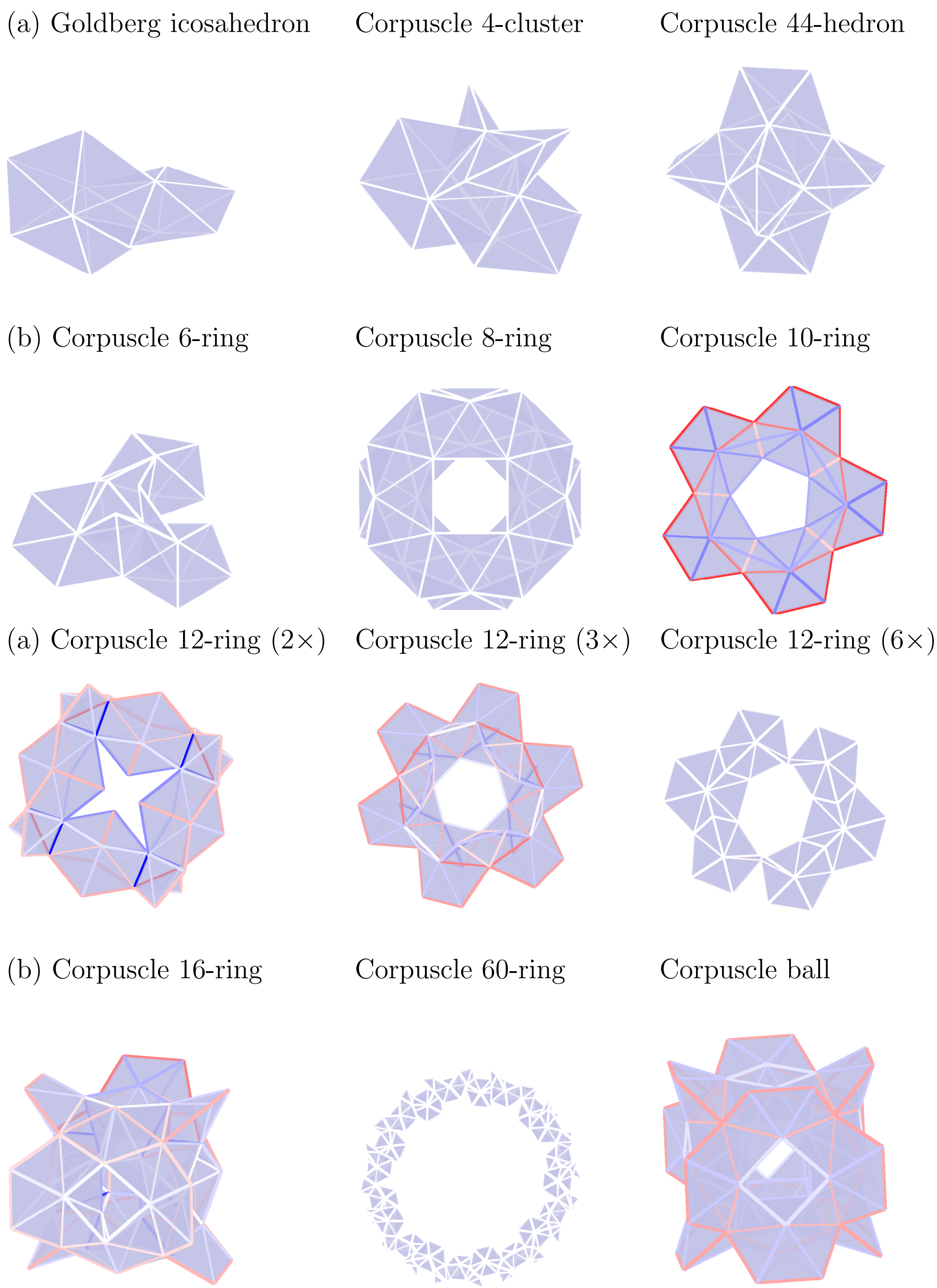

Figure 4: Edge tensions of corpuscle structures. In the relaxed states shown, edge lengths deviate from their natural value 1. Edge tensions are shown by colours (red: compression; blue: extension).

Acknowledgements

We thank Christoph Pöppe for contributing the 6-fold symmetric 12-ring.

References

[1] M. Goldberg: *Unstable polyhedral structures.* Mathematics Magazine **51** (3), 165–170 (1978).

[2] E. Wohlleben, W. Liebermeister: *The corpuscle — a simple building block for polyhedral networks.* Proc. 13th Internat. Conference on Geometry and Graphics, Dresden 2008.

[3] M. Thomas, J. Fruchterman, E.M. Reingold: *Graph drawing by force-directed placement.* Software – Practice and Experience **21**, 1129–1164 (1991).

[4] R.B. Fuller, E.J. Applewhite: *Synergetics. Explorations in the Geometry of Thinking.* Macmillan, New York 1975/1979.

A. Corpuscle structures studied

Here we summarise our results for the different corpuscle structures. The clusters (first three structures in Table 1 and Figures 2 and 4) can be built with rigid edges:

- As shown by Goldberg [1], the *Goldberg icosahedron* has three distinct conformations in which triangles are exactly regular. A paper model moves continuously and with little effort between these conformations. This "breathing" movement also shows up as the softest deformation mode in the calculations.
- The *four-corpuscle-cluster* consists of four elements, each containing four segments and one mouth. They are assembled in pairs around a non-regular octahedron just like the Goldberg icosahedron. According to the simulations, its softest deformation mode resembles the breathing of the Goldberg icosahedron, with the two halves opening and closing in opposite phase.
- The *44-hedron* contains six corpuscles with three segments and one mouth each. These "bridge" elements lean over the surface of a core solid, which resembles a regular icosahedron. In fact, it represents one phase of Buckminster Fuller's Jitterbug [4], a continuous movement between a regular octahedron, an icosahedron, and a cuboctahedron. The softest deformation mode of the 44-hedron consists of an extension along one of the main axes. Due to its symmetry, this mode can appear in three different directions.

The remaining structures are rings which all can be closed with no or little distortion. They show different symmetries, and some of them are slightly deformable.

- The *6-ring* emerges from alternating (2,2) and (0,4) elements. The (2,2) elements share the centre point of the ring as a vertex. In the paper model, the structure is flexible: its symmetric shape is unstable and flips into a conformation in which one of the (2,2) element diminishes its volume. So does the (0,4) element on the opposite side, while the other four elements simultaneously increase their volume. The vertical (2,2) element keeps some of its volume when the co-acting horizontal (0,4) element is already flat. This can happen in three different orientations. This behaviour is also reflected in the computer model: in the stable conformation, one of the vertical elements is punched in and completely flat, which decreases the overall tension to a very small value and breaks the 3-fold rotation symmetry.

- The *8-ring* consists of alternating elements of types (1,3) and (3,1) and is the narrowest ring that can be built from such elements. The empty area in its centre forms a square antiprism. The 8-ring can be closed without deformation and, as a paper model, is rigid. The *10-ring* consists of alternating elements of type (2,2) and (1,3). Five of its elements form a central pentagon. Its stable form is under tension and hardly flexible.

Our three 12-rings have the same number of nodes, edges, and triangles, but differ in their chain sequences and show 2-fold, 3-fold, and 6-fold symmetry, respectively.

- The *3-fold symmetric 12-ring* consists of alternating (1,3) and (3,1) elements. The inmost edges of six elements form a band meandering up and down three times. This band can be seen as part of a cube in the ring's empty centre. The structure closes with little tension and can be easily deformed, moving from the equilibrium conformation into three extreme conformations. In each of them, two (1,3) and two (3,1) elements become flat.
- The *6-fold symmetric 12-ring* is built from alternating elements of type (2,2) and (1,3) and surrounds a hexagon. In the paper model, the ring is harder to deform than the 3-fold-symmetric 12-ring. On the way from its symmetric shape to one of the three extreme positions, four of the elements become flat: two (2,2) elements and two (1,3) elements. After passing through a conformation of higher tension, the structure reaches a more relaxed shape. In the calculation, this ring undergoes a spontaneous symmetry breaking which leads to to the same shape: four segments, in a distance of three segments each, become almost flat and partially punched in.
- In the *2-fold symmetric 12-ring*, four elements of type (2,2) are assembled with four elements of type (1,3) and four elements of type (3,1). Deforming this ring creates extreme shapes (see Fig. 4(e)), since the co-acting sets of corpuscles contain elements of different types. One of the sets consists of four elements of type (2,2), and flattening these four causes the entire ring to contract. The second set consists of elements of type (1,3), and the third set of co-acting elements are type (3,1). A flattening of these sets causes a twist in the ring's shape.
- The *60-ring* is formed by a sequence of (1,3) type and (3,1) type elements which alternate after each third element. During the relaxation phase, the edge tension becomes very small. We therefore expect that the 60-ring can be built with edge lengths very close to the nominal length of 1.
- The *16-ring* consists of elements of types (1,3) and (3,1). It extends around a central empty cube, touching eight of the cube's edges and covering all its vertices. By adding four corpuscle elements as "bridges", it can be turned into a cube-centric corpuscle ball with cubic symmetry [2]. Both structures are hardly flexible.

Received August 6, 2010; final form June 25, 2012