Population Dynamics

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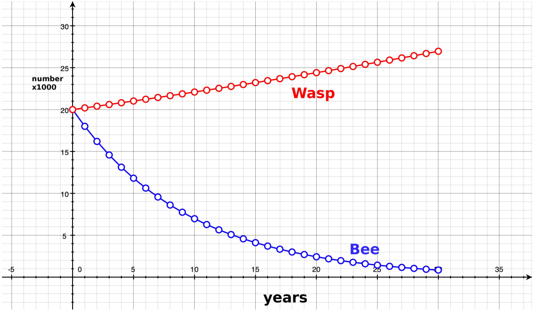
Discrete Population Models

- Differential equation models require overlap of generations
- Often no overlap between the generations (e.g. salomon, snowdrops, *Octopus Vulgaris*)
- Difference equation:
    - $N_{t+1} = f(N_t) = N_t F(N_t)$
- Discrete time steps
- Can in general be solved analytically

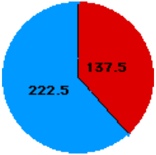
Small Example

- e.g. $N_{t+1} = rN_t$, $r > 0 \Rightarrow N_t = r^t N_0$
- Two populations
- Life span: 1 year
- Bees
  - 100 Bees produce 90 new ones each year $\Rightarrow r = 0.9$
- Wasps
  - 100 Wasps produce 101 new ones each year $\Rightarrow r = 1.01$



Fibonacci Sequence

- Rabbit population
  - Rabbits take one month to mature
  - Each productive pairs bears a new pair
  - Rabbits never die
- $R_{n+1} = R_n + R_{n-1}$
- Golden ratio: $\frac{N_t}{N_{t+1}} \approx \frac{\sqrt{5}-1}{2} \approx 0.618$
- Golden angle: $\frac{\sqrt{5}-1}{2} * 360 = 222.5 \Rightarrow \phi = 137.5$



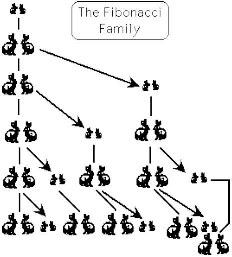


Figure: Rabbit pedigree taken from http://www.math.temple.edu/ reich/Fib/fibo.html

Fibonacci Sequence: Sunflower head

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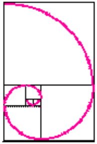
Fibonacci Sequence: Sunflower head

Further examples

picture from: www.world-myseries.com



picture from: wikipedia.org

picture from: www.mcs.surrey.ac.uk

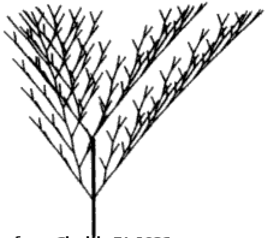


figure from: Girwish, T.J. 1986

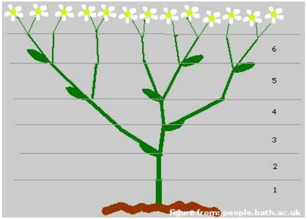


figure from: people.bath.ac.uk

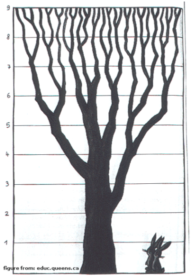
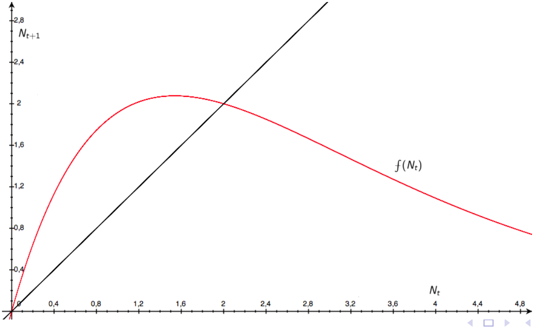


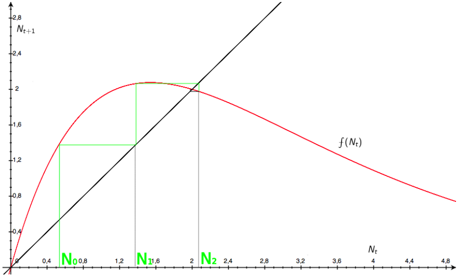
figure from: educ.queens.ca

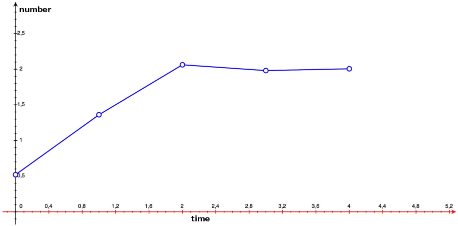
Analysing difference equations

- Remember: $N_{t+1} = f(N_t)$
- Steady state if $N_{t+1} = N_t$
- Example: $f(N_t) = N_t exp[r\left(1 - \frac{N_t}{K}\right)]$ (Ricker function)
  - $K \rightarrow$ max. capacity
  - $r \rightarrow$ intrinsic growth rate



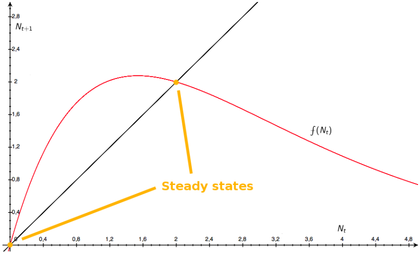
Graphical Solution: Cobwebbing



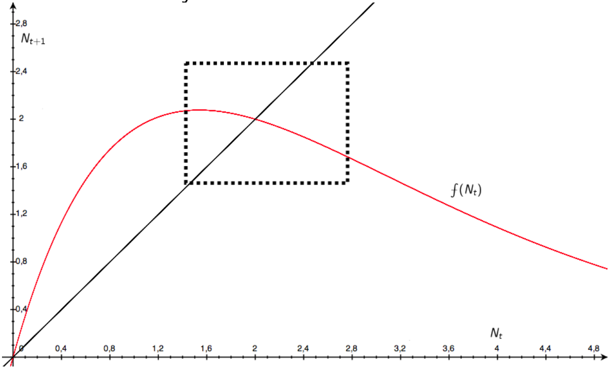


Steady State

- Steady state:
  Intersections of the the curve $N_{t+1} = f(N_t)$ and the line $N_{t+1} = N_t$
- Stable steady state: small pertubation $\Rightarrow$ system will fall back to steady state
- Derivative of $f(N_t)$ in the steady state determines stability

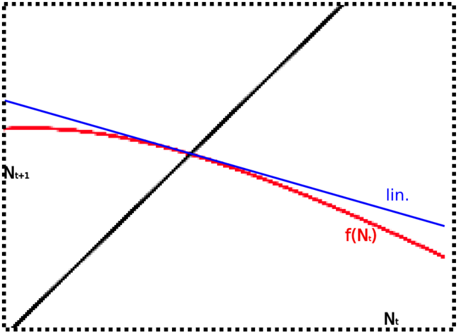


- Examine steady state in the example
- Linearize around steady state

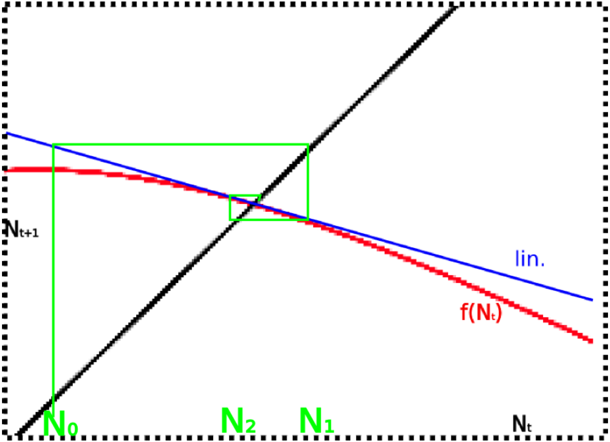


Linearization Around Steady State

- Derivation: $\frac{df}{dN_t}|_{N_t=N^*}$
- Linearization (1. taylor expasion)

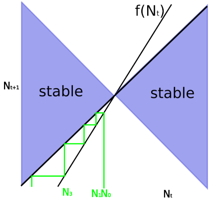


Stability: Graphical Solution

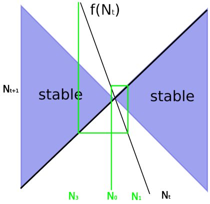


- This is a (linear) stable steady state

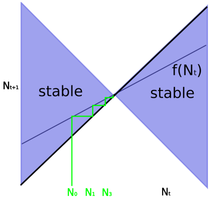
Stability: Graphical Solution



- slope > 1

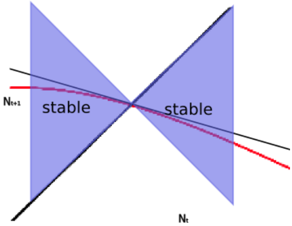


- slope < -1



- -1 < slope < 1

- Observation: Steady states with abs. derivative < 1 are stable



Stability and Summary

- Stability: Small perturbations vanish
- Discrete case $x_{t+1} = Ax_t$:
  - Solution: $x_t = A^t x_0$
  - $p(\lambda)$(max. abs. eigenvalue)$\leq 1 \Rightarrow$ Stability
- Continuous case $\frac{dx}{dt} = Ax$:
  - Solution: $x(t) = x_0 \cdot exp(tA)$
  - $v(\lambda)$(max. real part of eigenvalues)$\leq 0 \Rightarrow$ Stability

- Populations that reproduce in certain intervals $\rightarrow$ difference equations
- Steady state: intersection of $N_{t+1} = f(N_t)$ and $N_{t+1} = N_t$
- Fibonacci sequence in plants

Continuous Population Models

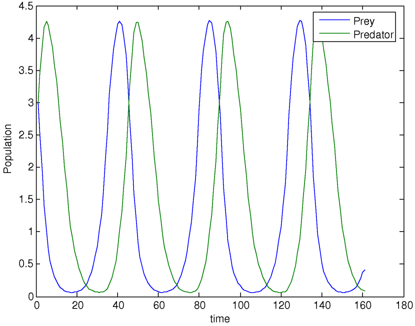
- Most populations have overlap between generations (e.g Humans)
- Can to some extend be modeled by ODEs
- $\frac{dS}{dt} = f(S, t)$
- Good for large populations

Predator-Prey: Lotka-Volterra Systems

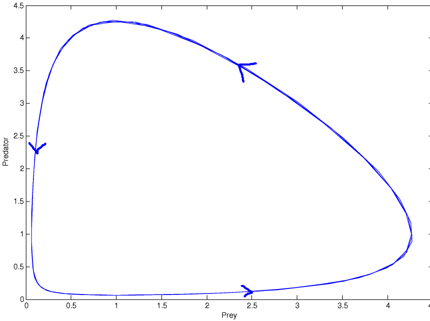
- Simple predator prey model
- $\frac{dN}{dt} = N(a - bP)$
- $\frac{dP}{dt} = P(cN - d)$
  - a growth rate prey
  - b neg. effect of predator on pey
  - c benefit of prey for pedator
  - d decay rate of predator
- prey in absence of predator grows unbounded
- predator reduces preys growth rate
- without prey the predator decays exponentially

Predator-Prey: Lotka-Volterra Systems

Timecourse



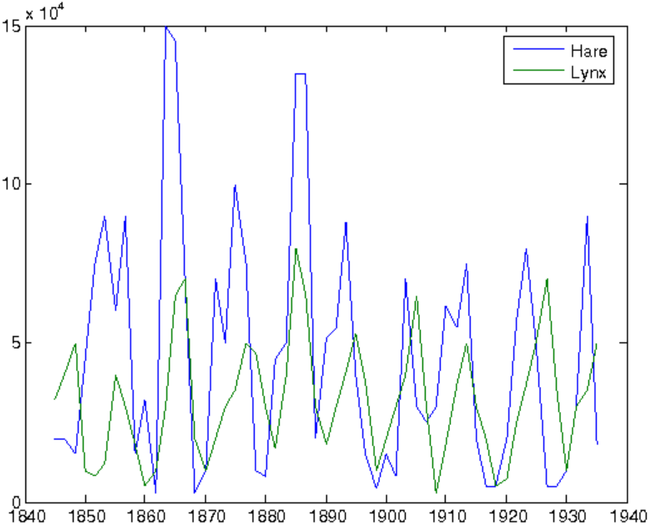
Phasediagram



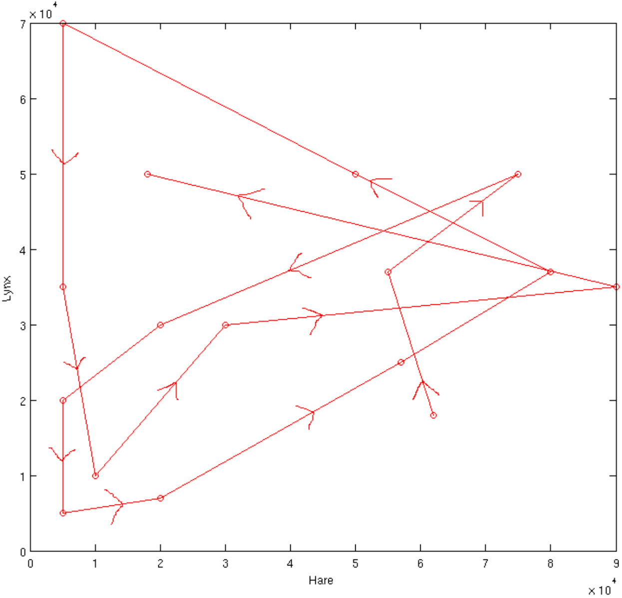
Example: Lynx - Snowshoe Hare

- Lynx hunt snow hares
- Long term data available (1845 - 1930)
- Data from fur catch records
  - Assumption: Fixed proportion of the population was caught

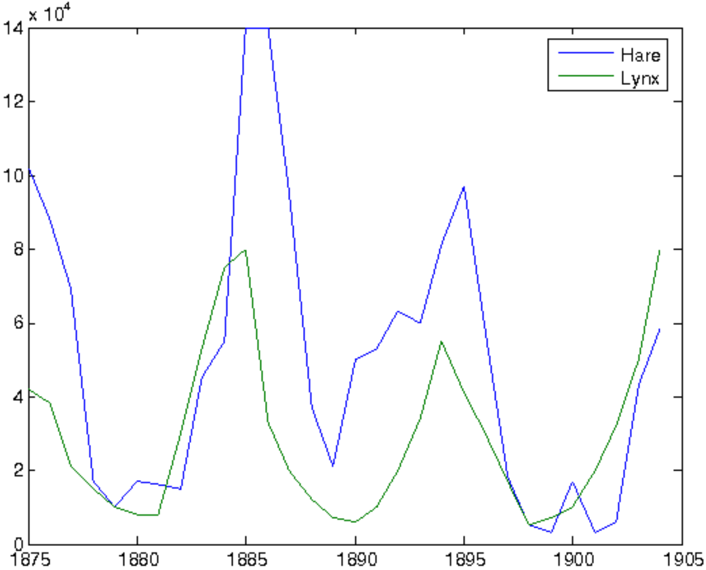
Timecourse Lynx Hare



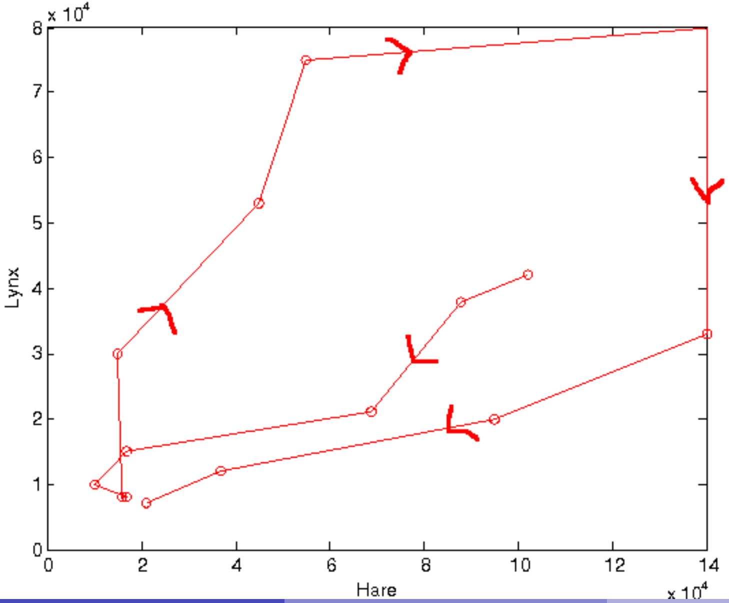
Phase Diagram 1910 - 1935



Time Course 1875 - 1904



Phase Diagram 1875 - 1904



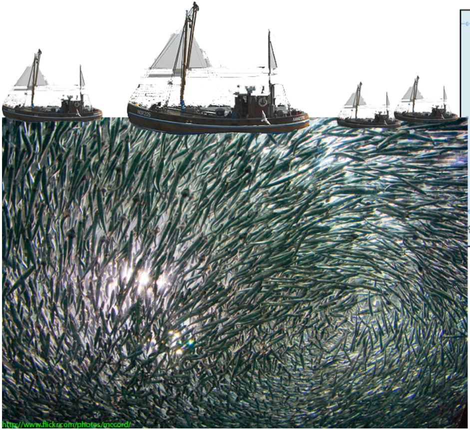
Does the hare eat the lynx?

- Proposal: The hare carries a disease
  - No such disease known
- Hunting is the disease
- Data does not represent a fixed proportion

Summary

- Non overlaping generations $\longrightarrow$ discrete models
  - Stability: max. abs. eigenvalue ≤ 1
- Difference equations occur in nature
- Overlap and continuous behaviour $\longrightarrow$ ODE models
  - Stability: max. real part of eigenvalue ≤ 0
- Modeling populations can involve traps

Peruvian Anchovies



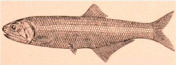
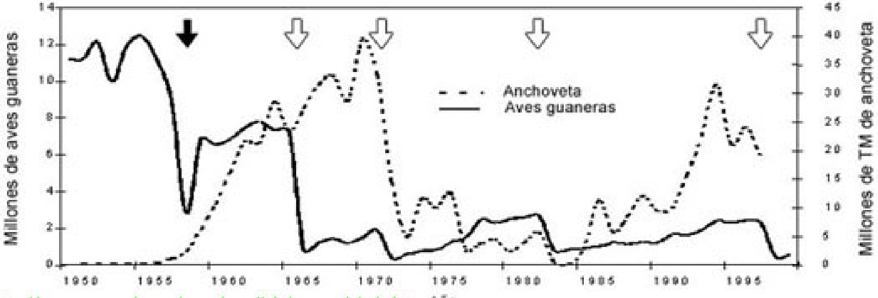


Figure: Fishing is a large economic sector in Peru

Peruvian Anchovies



http://www.mundoazul.org/english/guanobirds.htm

Figure: Sudden breakdown after many years of fishing.

Why?

Question: Why did this breakdown happen so sudden?

Why?

Approach: Mathematical modeling!

How to Model the Problem

What has to be taken into accont?

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Biology

- population size
- growthrate of anchovies
- amount of harvested fish

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Economy

- number of trawlers
- price for fish
- costs for fishing

Biology

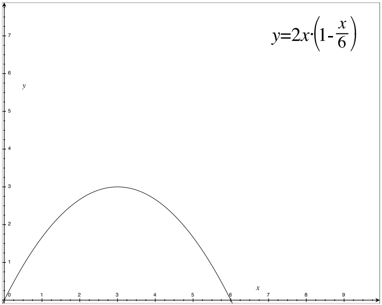
How can the growthrate of anchovies be modelled?

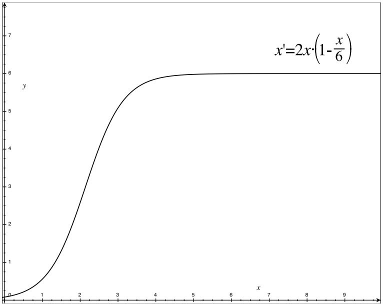
Biology

How can the growthrate of anchovies be modelled?

- We could use a simple *logistic equation* like this:

$$x' = rx(1 - \frac{x}{K}) \qquad (1)$$





Biology

- But:

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  - Anchovies live in huge swarms

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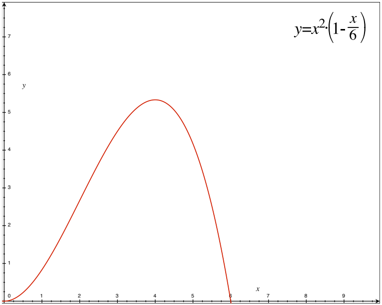
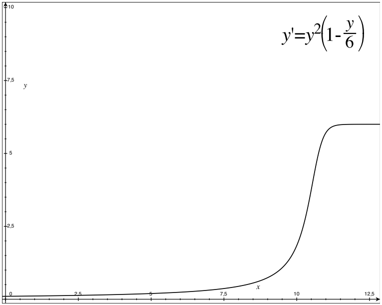
Biology

- But:
  - Anchovies live in huge swarms
  - The bigger the swarm the better the chances for survival and to find a partner
- So this function fits better:

$$x' = rx^2(1 - \frac{x}{K}) \quad (2)$$





Biology

- We add a small constant for fishes that can't be caught:

$$x' = a + rx^2(1 - \frac{x}{K}) \tag{3}$$

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- Finally we subtract the amount of fish that is harvested:

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- But how is E defined?

Economy

- We build a strongly simplified model of Economy:

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  - an abstract value for fishing effort E

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  - an abstract value for fishing effort E
  - fish has a price p
  - fishing has certain costs c
  - v is the fishing yield per inset unit E
- This yields the equation:

$$E' = \alpha E(pvx - c) \tag{5}$$

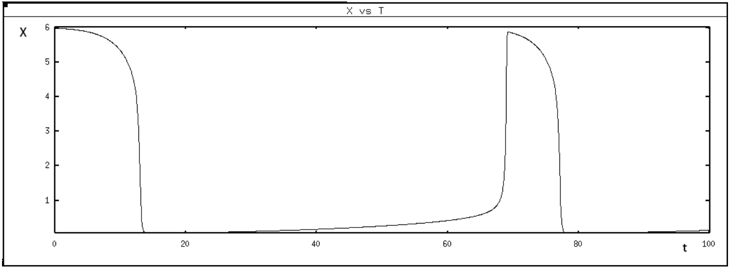
The final Model

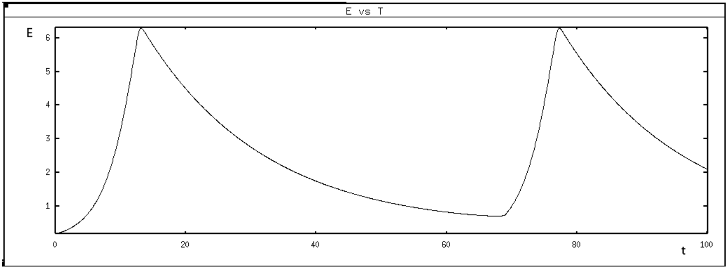
$$x' = a + rx^2(1 - \frac{x}{K}) - vEx \quad (6)$$

$$E' = \alpha E(pvx - c) \quad (7)$$

- a: small constant for remaining fishes
- r : linear growth factor
- K: unharvested equilibrium density
- v: gain per investement
- p: price for fishes
- c: cost per inset
- α: small factor to represent that E is a slow changing variable

Time Series





What is this Model good for?

1. We can analyse the behaviour of the model.
2. We can change parameters and get different results.
3. We can predict future developments and try to react before catastrophe happens.

Steady States

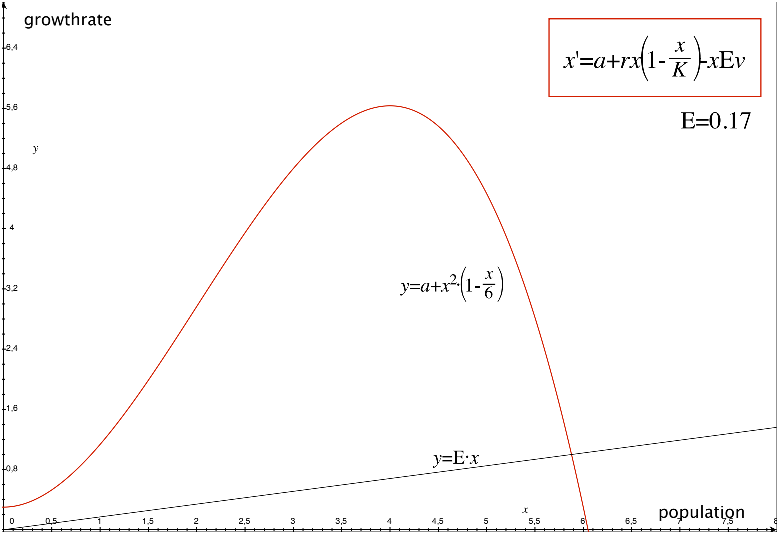


Figure: Stable steadty state at high population

Steady States

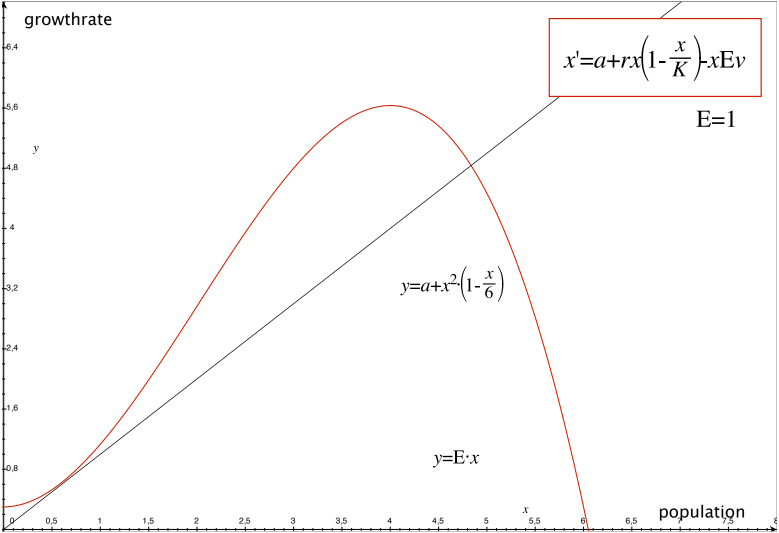


Figure: bifurcation point

Steady States

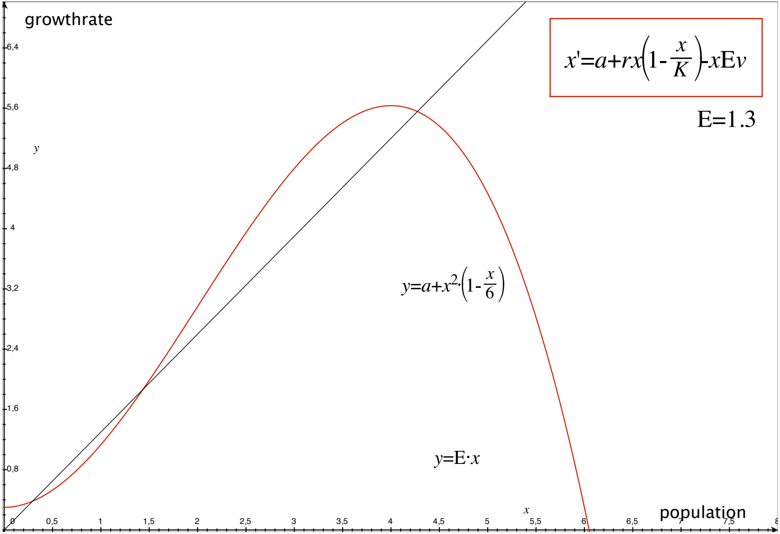


Figure: three steady states

Steady States

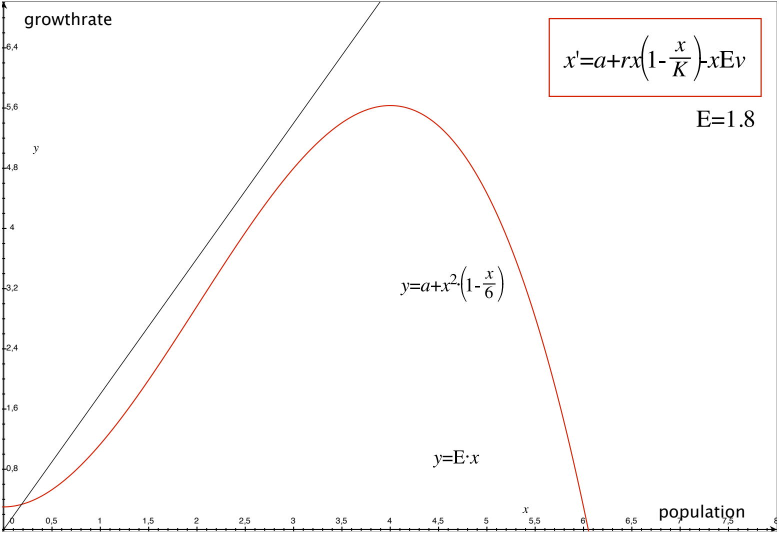


Figure: stable steady state at low level

Phaseplane

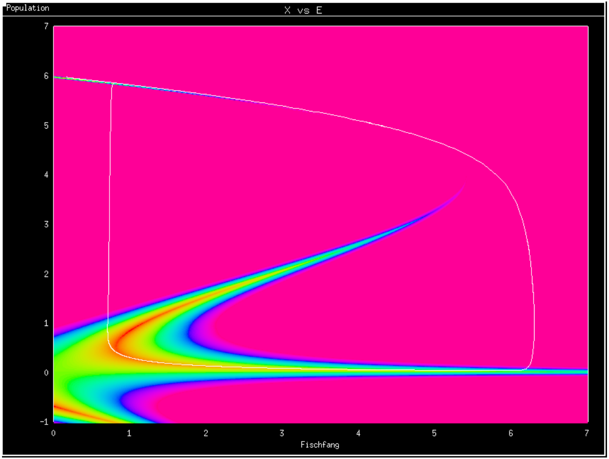


Figure: The Phaseplane with coloring by gradient

Hysteresis

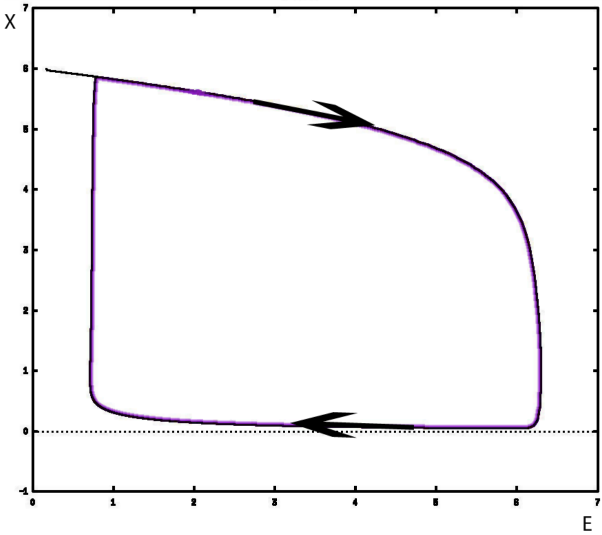
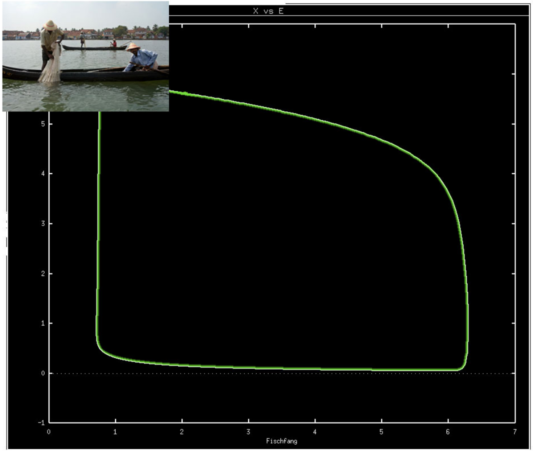
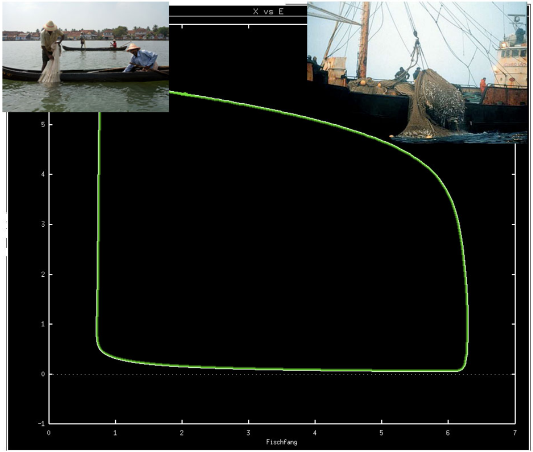


Figure: hysteresis - a memory of the system

Low Harvest

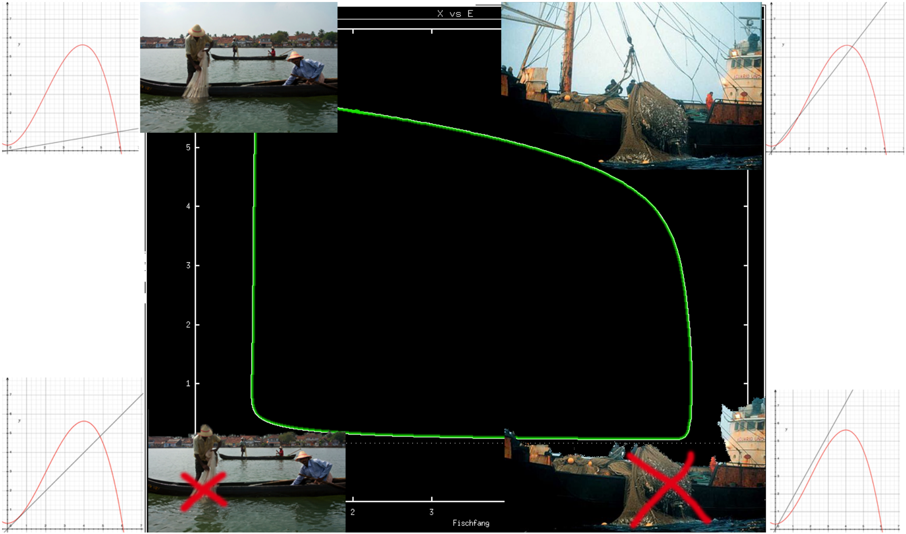


High Harvest



After Catastrophe

Regeneration



There are even worse cases!

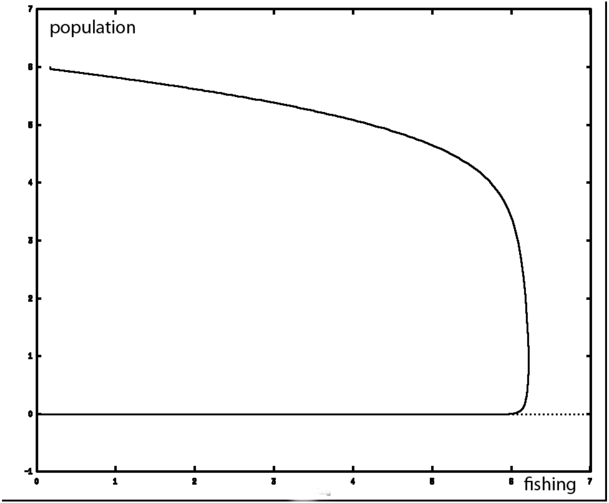


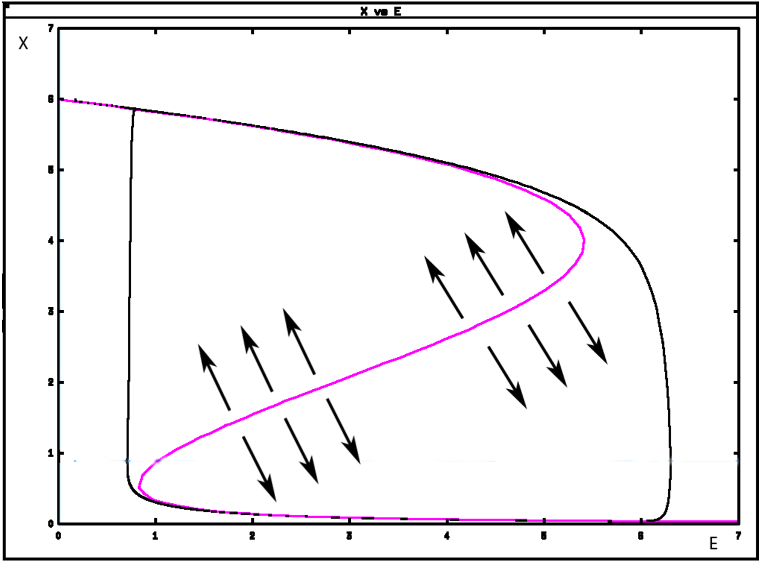
Figure: new fishes have to be added to the environment

Maximum Sustainable Yield

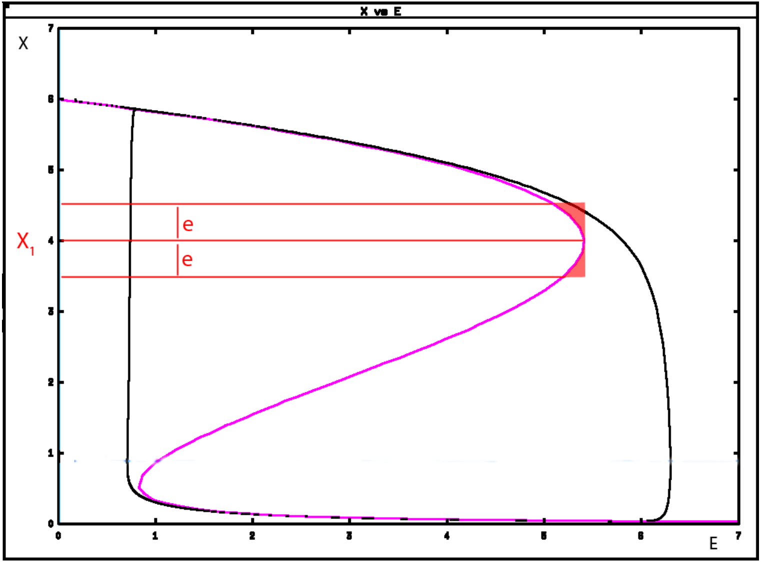
- maximum value for fishing before the breakdown occurs
- can be predicted, but should be used carefully
  - many different prediction methods with different outcomes
  - parameters are hard to determine
  - at MSY minimal disturbances can lead to breakdown of population

FOOD AND AGRICULTURE ORGANIZATION OF THE UNITED NATIONS
helping to build a world without hunger

Separatrix



El Niño



Outlook

WWF klagt gegen Kabeljau-Überfischung

Montag 19. März 2007, 12:12 Uhr

Hamburg (ddp-nrd). Die Umweltstiftung World Wide Fund für Nature (WWF) klagt seit Montag vor dem Europäischen Gerichtshof in Luxemburg gegen die Überfischung des Kabeljaus durch die Flotte der Europäischen Union. «Wir ziehen jetzt die juristische Notbremse gegen das andauernde Versagen der Fischereipolitik», sagte die Sprecherin der Organisation, Karoline Schacht, am Montag in Hamburg.

Nach Angaben des WWF ist die Zahl der geschlechtsreifen Dorsche in Nordsee, Irischer See, im östlichen Ärmelkanal und an der schottischen Westküste unter die Mindestmenge gefallen. Nach Artikel 7 des Kabeljau-Wiederaufbauplans müssten deshalb die Fangquoten um mehr als 15 Prozent gesenkt werden. Der EU-Fischereirat hat der Sprecherin zufolge jedoch nur eine Reduzierung der Fangquoten um 14 Prozent beschlossen. Die EU breche ihre eigenen Umweltgesetze, erklärte Schacht. Das dürfe nicht ungestraft geschehen.

«Den Zahlen nach geht es hier nur um wenige Prozentpunkte. Tatsächlich aber streiten wir für die Zukunft einer der ökologisch und wirtschaftlich wichtigsten Fischarten», betonte Schacht.

(ddp)